

A Novel Formulation of Baroclinic Rotational Equilibria

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We have developed a new formulation to obtain self-gravitating, axisymmetric configurations in permanent rotation. It is applicable not only to barotropic equations of state but also to baroclinic ones, for which angular momentum distributions are not cylindrical. The formulation is based on the Lagrangian variational principle. Some test calculations are presented, in which we have achieved an error of $O(10^{-4})$ in the Virial relation. We believe that this method could be a major breakthrough in stellar evolution theory, in which it is a common practice that rotation is included only approximately in one dimensional models.

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Self-gravitating bodies in permanent rotation have been studied for many years [1]. This is mainly because most of stars are approximately in such states and the study of rotational equilibria is important for the understanding of stellar structures. It is well known that uniform axisymmetric ellipsoids called the Maclaurin spheroids are one of such permanent rotators for incompressible fluids [2].

In the case of compressible fluids, however, such simple configurations are not known and numerical computations are required in general to obtain rotational equilibria with non-uniform stratifications. In particular, various techniques have been developed over the years for barotropic equations of state [3, 4], in which matter pressure is a function of density alone. This is because it is mathematically shown in this case that the rotational equilibria have cylindrical distributions of angular momentum, i.e. the angular momentum takes constant values on concentric cylinders and, as a consequence, there exists a first integral, the fact that expedites the numerical treatment enormously.

Stellar evolution theory is another natural application of rotational equilibria. Although the stellar evolution is a time-dependent problem, it is actually so slow in most stages compared with the dynamical time scale that it is well approximated by a collection of time-independent configurations in equilibrium with gradually changing parameters such as the central density. After the invention of the Henyey method [5], studies on the structures and evolutions of spherically symmetric stars in the absence of rotation (and/or magnetic fields) have been advanced exponentially over the years by many researchers[6].

In the presence of rotation, however, the same method cannot be employed. One of the difficulties in calculating rotational equilibria is the fact that we do not know a priori what the distribution of angular momentum looks like in the stellar interior. This is indeed the case even

for barotropes, in which the angular momentum distributions are cylindrical and the isopycnic and isobaric surfaces coincide with each other. It is hence a highly non-trivial issue how to make an appropriate sequence out of an ensemble of rotational configurations even if they are somehow obtained [7, 8].

In this letter, we propose a novel idea to obtain rotational equilibria numerically for both barotropes and baroclines[10], which is well suited for the application to the stellar evolution calculations. It is based on the Lagrangian variational principle: an energy functional is defined in such a way that its stationary point should correspond to a rotational equilibrium. It is then approximated on a triangulated mesh and minimized by moving grid points, the Monte-Carlo method quite analogous to what is commonly used in other fields, e.g. in nuclear physics to obtain deformed nuclei. We demonstrate its nice performance in some test models.

Lagrangian variational principle.—Our formulation is based on the Lagrangian variational principle, in which rotational configurations are obtained by minimizing the following functional under the conditions stated below:

$$E[\xi(\mathbf{r})] = \int \varepsilon \rho dV + \frac{1}{2} \int \rho \phi dV + \int \frac{1}{2} \rho \left(\frac{j}{\varpi} \right)^2 dV. \quad (1)$$

In this expression, ϖ is the distance from the rotation axis and dV stands for the volume element; the integration domain is the stellar interior; ρ , ε , ϕ and j denote the density, specific internal energy, gravitational potential and specific angular momentum, respectively. They are regarded as functionals of the so-called Lagrangian displacement vector $\xi(\mathbf{r}) \equiv \mathbf{r}'(\mathbf{r}) - \mathbf{r}$, which connects the fluid element at \mathbf{r} in an arbitrarily chosen reference configuration to the corresponding fluid element shifted to \mathbf{r}' in the configuration obtained by the deformation of the reference. Then a rotational equilibrium is obtained from

a deformation or ξ that corresponds to a stationary point of the functional $E[\xi]$ given by $\delta E[\xi]/\delta \xi = 0$. In taking this Euler-Lagrange derivative, the specific entropy (denoted by s hereafter) and angular momentum are fixed for each fluid element, which is why this formulation is referred to as *Lagrangian*.

Note that the Lagrangian displacement vectors themselves are not necessarily small. The density as a functional of ξ is given by mass conservation: $\rho(\mathbf{r}') = J(\mathbf{r})\rho_0(\mathbf{r})$. Here ρ_0 is the density distribution in the reference configuration and $J(\mathbf{r})$ is the Jacobian at \mathbf{r} for the transformation $\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \xi(\mathbf{r})$. Since the specific internal energy is a function of density and specific entropy and the latter is unchanged by deformation, ε can be regarded as a functional of ξ . The gravitational potential is related with the density as $\Delta\phi = 4\pi G\rho$, which makes it possible to consider it also as a functional of ξ [11]. Here G is the gravitational constant.

A finite element method is adopted in our implementation of the Lagrangian variational principle presented above: the meridian section of an axisymmetric star is covered by a triangulated mesh, and its grid points or nodes are moved to deform the star; all physical quantities are assigned to the nodes; in particular, the mass, specific entropy and angular momentum allotted to each node are fixed in the node shifts; the energy functional then becomes approximately a function of the coordinates of nodes and its minimization gives a rotational equilibrium for given distributions of mass, specific entropy and angular momentum on the triangulated mesh. It is emphasized that the variables to be solved in this formulation are the coordinates of each grid point unlike in the common Eulerian formulations, in which values of physical quantities on fixed grid points are quested for.

The approximate energy functional is given as follows:

$$E_{\text{FEM}}(\mathbf{r}_i) = \sum_i \varepsilon_i m_i + \frac{1}{2} \sum_i \phi_i m_i + \sum_i \frac{1}{2} \left(\frac{j_i}{\varpi_i} \right)^2 m_i, \quad (2)$$

which is now actually a function of the coordinates of all nodes, \mathbf{r}_i , with the subscript specifying the individual node. In this expression m_i denotes the mass assigned to the i -th node, which is also constant just like the specific entropy and angular momentum. The density at the i -th node is expressed as $\rho_i = m_i/V_i$, in which V_i is a volume allotted to the i -th node and evaluated as a function of the coordinates of nodes from the volumes of the cells attached to this node. It is then obvious that ε_i is also a function of the locations of nodes. As for the gravitational potential ϕ_i , the Poisson equation is solved numerically on the same triangulated mesh for the above ρ_i , which makes it possible for us to regard ϕ_i as a function of node positions again. With the energy function at hand, all we need to do is to minimize it with respect to the node coordinates for a given triplets of (m_i, s_i, j_i) .

Minimization.— There are two possibilities to ob-

tain a rotational equilibrium that corresponds to the minimum of the energy function given by Eq. (2). The first choice is that one takes its derivatives with respect to \mathbf{r}_i to generate a system of coupled nonlinear equations for \mathbf{r}_i and solve it numerically. The resultant equations can be regarded as an approximate expression of the force balance among gravity, pressure and centrifugal force. The second option is to randomly shuffle the coordinates of the grid positions and search for the configuration that gives the minimum to the energy function. In our formulation we choose the latter. There are a couple of reasons for this choice, which will be detailed in our forthcoming paper, but the most serious one is that the coupled nonlinear equations for force-balance are difficult to solve numerically by the Newton-Raphson method owing to the existence of small eigenvalues.

The actual minimization procedure is the following. Given an initial configuration, we move one of the nodes in the radial direction slightly and see if this shift lowers the value of the energy function or not. If not, we cancel the shift. The amount of dislocation is a random variable but is limited to $\sim 1\%$ of the lengths of the edges attached to the node. We do the same things in the lateral direction and sweep the entire mesh. We repeat this process until the value of the energy function is no longer lowered by a certain amount ($\sim 1\%$). The nodes on the outer boundary, which are referred to as the anchor nodes and have a vanishing mass, are fixed in this process, since an expansion in the radial direction would continue for ever otherwise. In some cases, however, the outer boundary comes too close to or goes too far from the nearby active nodes. We then shift the anchor nodes appropriately and fix them thereafter. We find that large deformations of triangular cells degrade accuracy of the energy function and in some cases generate a fictitious local minimum. To avoid such artifacts we apply a smoothing to the deformed portion of the grid whenever we detect too acute ($\lesssim 10^\circ$) an angle in triangles.

Test calculations.— We apply our new method to simplified models of main-sequence stars with rapid rotations. We assume that the stars are composed of hydrogens alone and employ an ideal-gas EOS. We first consider an isentropic model as a representative barotropic case. We assign an identical specific entropy to all nodes, $s_i = s_0$ with $s_0 = 15.0k_B$. Here k_B is the Boltzmann's constant. The initial configuration is a spherical star of $0.60M_\odot$ with M_\odot being the solar mass. It is obtained by solving the Lane-Emden equation for the same specific entropy and mapping the result on a triangulated mesh with 852 nodes. The mass on each node m_i is obtained from the density and volume assigned to the node in the initial configuration. As for the angular momentum, we assume in the initial spherical configuration an angular velocity distribution of the following form: $\Omega(\mathbf{r}) = \Omega_0 \varpi_0^2 / (\varpi^2 + \varpi_0^2)$. Here ϖ_0 and Ω_0 are model pa-

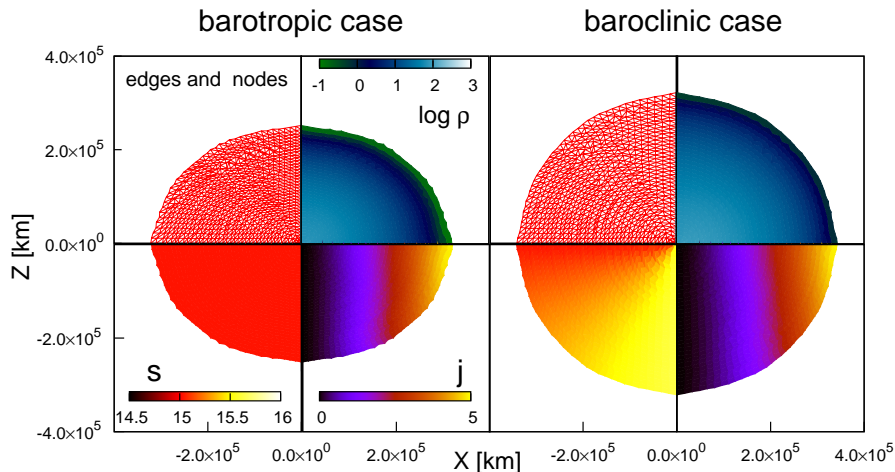


FIG. 1: (color on line). Structures of a star in rotational equilibria for barotropic (left four panels) and baroclinic (right four panels) EOS's. The upper left quadrants show the nodes and edges in the triangulated mesh. The other panels display clockwise the color contours of logarithmic density in g/cm^3 , specific entropy in k_B and specific angular momentum in $10^{18}\text{cm}^2/\text{s}$. The color scales are identical for both cases.

rameters and the former is set to the radius of the spherical star whereas the latter is $\Omega_0 = 10^{-6}\pi \times 5000 \text{ rad/s}$, which is roughly 5000 times the solar angular velocity.

In this model, the rotational equilibrium is obtained after 525 sweeps when the value of the energy function does not change by more than 1% in the last 50 sweeps. The results are given in the left four panels of FIG. 1. In the upper left quadrant the mesh configuration is displayed by nodes and edges. The distributions of density and specific entropy are shown in the upper right and lower left quadrants, respectively. Since this is an isentropic model, the specific entropy is constant in the stellar interior. From these pictures, it is evident that the rotational equilibrium is oblate, which is realized by lateral motions of nodes in our formulation. In the lower right quadrant, we show the distribution of the specific angular momentum. It is clear that the cylindrical rotation is realized as it should be. It is emphasized that this is not a trivial thing. In fact, the rotation law is not known a priori in our formulation. The cylindrical rotation is *not assumed but obtained* as a result of computations. This is a strong vindication that our formulation works well indeed.

Now we proceed to a baroclinic case. As an initial configuration we employ not a spherical star by the the barotropic equilibrium just obtained. This is meant to observe clearly how each node moves as a response to the introduction of baroclinicity. It is also helpful in terms of the numerical cost. We have confirmed that the formulation works equally well for an initially spherical configuration. The baroclinicity is most easily introduced by the artificial modification of the entropy distribution. Here we do this in the following way. Since we are using the ideal-gas EOS and the specific entropy is proportional to $K = P/\rho^{5/3}$, we modify the value of K on each node ac-

cording to the relation: $K_i = K_0(1 + \theta_i/\pi)$, in which K_0 is the value of K that gives s_0 in the barotropic model, and θ_i is the latitude of node i . This entropy distribution is admittedly quite artificial and will be much different from reality (see e.g. [6]) and it should be understood that this is just for illustrative purposes. Note, however, that the resultant configuration is stable against convection by the Høiland criterion.

We are again successful to obtain a rotational equilibrium also for this model after 255 sweeps, at which point the energy function does not change by more than 1% in the last 50 sweeps. The results are displayed in right four panels of FIG. 1. The grid configuration, density, specific angular momentum, specific entropy are shown in each quadrant clockwise from the upper left corner. It is clear from these pictures that the baroclinic configuration is much less oblate compared with the barotropic counterpart. This is due to the enhanced thermal pressure near the rotation axis, which leads to the expansion of the star in the direction parallel to the axis. The baroclinicity of the model is recognized most easily by the comparison between the distributions of density and specific entropy, since the isopycnic surfaces should coincide with the iso-entropic surfaces in the barotropic model, which is certainly not the case in this model. The baroclinicity of the model is also reflected in the angular momentum distribution, which is obviously *not* cylindrical.

In the upper panel of FIG. 2 we show the histories of the values of the energy function in the minimizations for both the barotropic and baroclinic models. The result for the latter is attached at the end (the 525th iteration) of the result for the former. Except for a big jump at the junction point, the value of the energy function decreases most of the time. There are some small

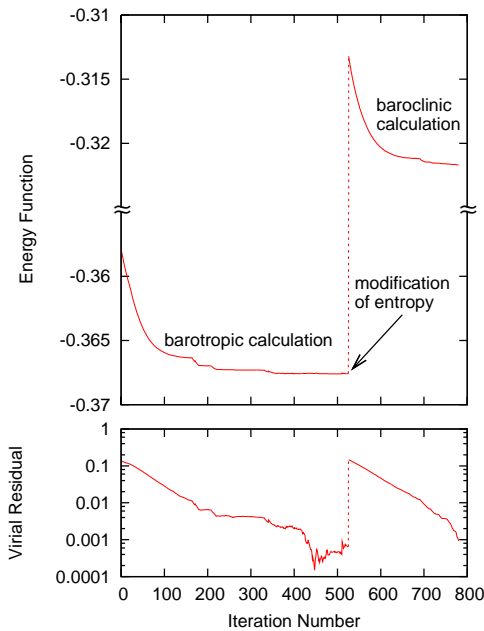


FIG. 2: The values of the energy function and of the virial residual in the minimization. The baroclinic case is attached at the end of the barotropic case as indicated by the arrow. The energy function is normalized by GM_{\odot}^2/R_{\odot} , in which R_{\odot} is the solar radius.

glitches occurring sporadically, however. They correspond to the smoothings, which are applied when a triangular cell is deformed too much and increase the value of the energy function inevitably. These smoothings are necessary to ensure convergence to the true minimum, which is indeed observed in FIG. 2. Also shown in the lower panel is the residual in the virial relation defined as $V_C = |(2T + W + 3\int P dV)/W|$, in which T , W are the rotational and gravitational energies, respectively, and the third term is the volume integral of pressure. This is a measure of numerical accuracy commonly used in the literature. The results of both models are shown consecutively again. It is clear that $V_C \sim O(10^{-4})$ is achieved in both cases. It should be noted that this accuracy is attained with 852 nodes in this study.

Discussions.— It is emphasized that the establishment of the formulation that can handle not only barotropic but also baroclinic rotational equilibria is itself a major achievement, since such attempts are not many in the literature [7, 8] and all previous formulations are Eulerian. It will not be a simple task to apply them to stellar evolution calculations. In sharp contrast, our formulation is Lagrangian and we can trace the movement of each fluid element during the quasi-static stellar evolutions. Combined with the network calculations of nuclear reactions at each node, our new formulation will be able to treat the stellar evolution rather easily. Radiative energy transport will be handled by solving a diffusion

equation on the triangulated mesh. Angular momentum transfer may be also approximated as diffusions just as in the current stellar evolution models.

The biggest challenge is the treatment of convection. Whereas the energy functional is *minimized* in our formulation, convectively unstable equilibria do not correspond to the minimum but to a stationary point (or a saddle point) of the energy functional. A naive application of the formulation to such configurations will lead to a continued distortion of the mesh in the convection zone. A couple of ideas are currently being tested and will be reported elsewhere. Appropriate triangulations, which are not touched in this letter, will become important in the application of the method to stars in advanced evolutionary stages, in which the stars develop a core and mantles with large density contrasts. It may be necessary to extend the formulation so that it could handle multi-layers. Possible applications are not limited to these stars but will be extended to e.g. compact stars, proto-stars and planets. Incorporation of magnetic fields and/or general relativity should be considered in due course. Our new formulation will be a real break-through then.

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